ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, Nos. 1-2, 1978

JONSSON ALGEBRAS IN SUCCESSOR CARDINALS

BY

S. SHELAH[†]

ABSTRACT

We shall show here that in many successor cardinals λ, there is a Jonsson algebra (in other words $\mathrm{Jn}(\lambda)$, or λ is not a Jonsson cardinal). In connection with this we show that, e.g., for every ultrafilter D over ω, in $(\omega_{\omega}, <)^{\omega}/D$ there is no increasing sequence of length $\aleph_{(2^{\aleph_0})^+}$. On Jonsson algebras see e.g. [1]; for successor $\lambda^+ = 2^{\lambda}$ there is a Jonsson algebra, $\mathrm{Jn}(\lambda) \Rightarrow \mathrm{Jn}(\lambda^+)$ (due to Chang, Erdös and Hajnal) and even in $2^{\aleph_\alpha} = \aleph_{\alpha+n}$ ([3]). We give here a method to prove, e.g., $\mathrm{Jn}(\aleph_{\omega+1})$ when $2^{\aleph_0} \leqq \aleph_{\omega+1}$ and $\mathrm{Jn}(2^{\aleph_0})$ when $2^{\aleph_0} = \aleph_{\alpha+1}, \alpha < \omega_1$; and similar results for higher cardinals.

Questions. (1) Does $\mathrm{Jn}(\aleph_{\omega+1})$ always hold?

(2) Does $\mathrm{Jn}(\lambda^+)$ always hold, or at least when $(\lambda^+)^{\aleph_0} = \lambda^+$?

(3) Does always $\aleph_{\omega+1} \in P\,\mathrm{cf}\langle \aleph_n : n < \omega \rangle$?

Definition 1. (A) A Jonsson algebra is an algebra M, with countably many operations (finitary, of course), which has no proper subalgebra of the same cardinality. A Jonsson model is a model with countably many relations and operations which has no proper elementary submodel of the same cardinality.

(B) $\mathrm{Jn}(\lambda)$, or λ is not a Jonsson cardinal if there is a Jonsson algebra of cardinality λ. This is equivalent to the existence of a Jonsson model (expand by Skolem functions).

Convention 2. (A) We do not distinguish between a model and its universe; and unless stated otherwise a model has only countably many operations and relations.

(B) For simplicity we restrict ourselves to models of the form M_λ, where M^1_λ will be $(H(\lambda^*), \in)$ for $\lambda^* > \lambda$ (e.g. $(2^{\lambda})^+$) ($H(\lambda^*)$ is the family of sets whose

[†]The author would like to thank the United States—Israel Binational Science Foundation for partially supporting his research by grant 1110.

Received December 5, 1976 and in revised form May 25, 1977

transitive closure has cardinality $<\lambda^*$); let M_λ^2 be an elementary submodel of M_λ^1 of cardinality λ, $\lambda+1 \subseteq M_\lambda^2$, and $M_\lambda = (M_\lambda^2, \in, F)$ where F is a one-to-one function from λ onto M_λ^2. So M will denote some M_λ.

Notice that $\mathrm{Jn}(\lambda)$ implies that any M_λ is a Jonsson model (proof as for 4A). If there is a Jonsson algebra $\mathfrak{A} = (\lambda, f_i)_{i\in\omega}$ then $\mathfrak{A} \in M_\lambda^1$, thus $M_\lambda^1 \models$ "there is a Jonsson algebra on λ". By way of contradiction, assume there is a $N < M_\lambda$, $N \neq M_\lambda$, $\|N\| = \lambda$. Clearly (since λ is definable in M_λ as $\sup \mathrm{Dom}\, F$) $\lambda \in N$ and $N \models$ "there is a Jonsson algebra on λ". Let $\mathcal{B}$ be such an algebra but $\mathcal{B} \cap N < \mathcal{B}$, $\mathcal{B} \cap N \neq \mathcal{B}$ (for $\lambda \not\subseteq N$) and $\|\mathcal{B} \cap N\| = \|\lambda \cap N\| = \lambda$. This is a contradiction to $\mathcal{B}$ being Jonsson.

Definition 3. (A) For sets S_1, S_2 of cardinals, and a cardinal (or ordinal) μ, $S_1 \to S_2[\mu]$ means that for every M (as in 2B) and $N < M$, *if*

(i) $\mu + 1 \subseteq N$ (for $\mu = \aleph_0$ this is empty),

(ii) for every $\lambda \in S_1$, $|\lambda \cap N| = \lambda$,

(iii) $S_1 \subseteq N$ (if each $\lambda \in S_1$ is a successor, this follows by (ii)),

(iv) $S_1, S_2 \in N$,

then for some $\lambda \in S_2$, $|\lambda \cap N| = \lambda$ and $\lambda \in N$. (The interesting case is $\mathrm{Sup}\, S_1 \geqq \mathrm{Sup}\, S_2 + \mu$.)

(B) When $S_l = \{\lambda\}$ we write λ instead of S_l, and instead of $S_l^1 \cup S_l^2$ we write S_l^1, S_l^2. Note that in 3(A) we can replace S_l by a sequence, and nothing changes.

For Notational simplicity let $\mathrm{Sup}\, S = \bigcup\{\lambda + 1 : \lambda \in S\}$.

Observation 4. (A) $S_1 \to S_2[\mu]$ iff (*) iff (**), where

(*) There is a model N_0, $\mathrm{Sup}\, S_1 \subseteq N_0$, N_0 has $\leqq |\mu|$ operations and relations and if $N < N_0$, $|N \cap \lambda| = \lambda$, $\lambda \in N$ for each $\lambda \in S_1$ then $|N \cap \lambda| = \lambda$, $\lambda \in N$ for some $\lambda \in S_2$.

(**) There is a model N_0 as in (*) with universe $\mathrm{Sup}\, S_1$.

(B) In Definition 3A(i) we can demand only $\mu \subseteq N$ or even $|\mu| \subseteq N$ for μ ordinal.

(C) In Definition 3A we can demand M to vary only on $M_\lambda < H(\lambda^*)$ where $\lambda = \mathrm{Sup}\, S_1$ and $\lambda^* > \lambda$ is a constant, and demand some specific elements $\in M_\lambda$.

Proof. $S_1 \to S_2[\mu] \Rightarrow$ (*): take $\lambda = \mathrm{Sup}\, S_1$, $N_0 = (M_\lambda, S_1, S_2, i)_{i \leq \mu}$.

(*) $\Rightarrow$ (**): take N_0 as in (*). Since any $N_1 < N_0$ s.t. $\mathrm{Sup}\, S_1 \subseteq N_1$ satisfies (*) we can assume $\|N_0\| = \mathrm{Sup}\, S_1$. Add Skolem functions to N_0 and add a name to each formula, getting a model N_1 satisfying (*). Take $N_2 = N_1 \restriction \mathrm{Sup}\, S_1$. We show N_2 satisfies (**). Let $N_2' < N_2$ such that $(\forall \lambda \in S_1)(\lambda \in N_2' \wedge |\lambda \cap N_2'| = \lambda)$; take N_0'—the Skolem closure of N_2' in N_0. By (*) for N_0 there is $\lambda \in S_2$ s.t. $\lambda \in N_0'$ and

$|\lambda \cap N_0'| = \lambda$. Since $|N_0'| \cap \operatorname{Sup} S_1 = |N_2'|$ we have $\lambda \in S_2$ s.t. $\lambda \in N_2'$ and $|\lambda \cap N_2'| = \lambda$.

(**) $\Rightarrow$ $S_1 \to S_2[\mu]$. Suppose N_0 is as in (**), and with minimal μ (for the given S_1, S_2); hence $\mu \in M_\lambda$. Suppose $N < M_\lambda$, as in 3(A). Now $N_0 \in M_\lambda^1$, but as $M_\lambda^2 < M_\lambda^1$, w.l.o.g. $N_0 \in M_\lambda^2$, and even $N_0 \in N$. So N_0^*, the submodel of N_0 with universe $N_0 \cap N = \{a : N \vDash \text{“}a \in N_0\text{”}\}$, has universe $N \cap \operatorname{Sup} S_1$ and $N_0^* < N_0$.

By the hypothesis of 3(A), the hypothesis of (*) holds, so for some $\lambda \in S_2$, $\lambda \cap N_0^*| = \lambda \in N_0^*$ hence $|\lambda \cap N| = \lambda \in N$, so we finish.

(B), (C) Easy from (A).

The basis of our proof is the following

Observation 5. (A) If $\lambda \to \mu^+[\aleph_0]$ for every $\mu < \lambda$, then $\mathrm{Jn}(\lambda)$.

(B) If $\aleph_\alpha \to \mu^+[\mu]$ for every $\mu < \aleph_\alpha$ and $\alpha \subseteq N < M_{\aleph_\alpha}, \|N\| = \aleph_\alpha$ then $N = M_{\aleph_\alpha}$.

(C) If $N < M_\lambda, \|N\| = \lambda$, and for each $\mu \in N, \mu < \lambda, |N \cap \mu^+| = \mu^+$ *then* $N = M_\lambda$.

(D) If $\mathrm{Jn}(\lambda)$, then $\lambda \to \kappa[\aleph_0]$ for every $\kappa \leqq \lambda$.

Proof. (A) By (C); let $N < M, \|N\| = \lambda$, now $\mu \in N$ implies $\mu^+ \in N$, so by a hypothesis $|N \cap \mu^+| = \mu^+$.

(B) Like (C), as for $\mu < \lambda, \mu = \aleph_\beta$ for some $\beta < \alpha$ hence $\mu \in N$.

(C) Because of the function F it suffices to prove $\lambda \subseteq N$, and we know $|N \cap \lambda| = \lambda$.

Let μ be a maximal cardinality for which $\mu \subseteq N$. If $\mu = \lambda$ we finish, and if $\mu \in N$ then by a hypothesis $|N \cap \mu^+| = \mu^+$, but then $\mu^+ \subseteq N$ (there is $f = f^\mu \in N$, such that for every $\beta < \mu^+, x \mapsto f(\beta, x)$ is a map from μ onto β; so for each $\alpha < \mu^+$, there is $\beta \in N, \alpha < \beta < \mu^+$, so for some $\gamma < \mu, f(\beta, \gamma) = \alpha$, hence $\alpha \in N$). So $\mu \notin N$. Choose a minimal $\alpha, \mu \leqq \alpha \in N$; as $|\alpha| \in N, \alpha$ is a cardinal. Clearly $\alpha < \lambda$ (as $\|N\| = \lambda$, and by F) so $|\alpha|^+ \in N$, hence $|N \cap |\alpha|^+| = \alpha|^+$, so for some $\gamma \in N, \alpha < \gamma < |\alpha|^+, |N \cap \gamma| = |\alpha| > \mu$, using $f^{|\alpha|}(\gamma, x)$ we get a contradiction.

(D) By 4(*).

Lemma 6. (A) *If* $S_0 \to S_1[\mu]$, *and for each* $\kappa \in S_1$ $S_0, \kappa \to S_2[\mu]$ *then* $S_0 \to S_2[\mu]$.

(B) *If* λ_i $(i \leqq \alpha)$ *is an increasing sequence of cardinals, and* $\lambda_i \to \{\lambda_j : j < i\}[\mu]$ *then* $\lambda_\alpha \to \lambda_0[\mu]$ *(we can replace the assumption by: for every i for some nonempty* $S_i \subseteq \{\lambda_i : j < i\}, \lambda_i \to S_i[\mu]$*).*

(C) *The relation* $S_1 \to S_2[\mu]$ *is preserved under increasing* S_1, S_2 *and* μ.

PROOF. (A) By 4(*) there is a model on $\lambda = \operatorname{Sup} S_0$ with $\leqq \mu$ relations demonstrating that $S_0 \to S_1[\mu]$. Add to this model μ relations demonstrating for every $\kappa \in S_1$: $S_0, \kappa \to S_2[\mu]$. The resulting model shows $S_0 \to S_2[\mu]$.

(B), (C) Similar proofs.

By 5 and 6(B), in order to prove the existence of Jonsson algebras it suffices to prove enough cases of the form $\lambda \to S[\aleph_0]$.

LEMMA 7. (A) $\lambda^+ \to \lambda[\aleph_0]$ (*hence by* 6(A) $\aleph_{\alpha+n} \to \aleph_\alpha[\aleph_0]$).

(B) $\lambda \to \operatorname{cf}\lambda[\aleph_0]$.

(C) $2^\lambda \to \lambda[\aleph_0]$ *when* $2^\mu < 2^\lambda$ *for every* $\mu < \lambda$.

(D) $\lambda \to \{\lambda_i: i < \delta\}[\delta]$ *if* $\lambda_i < \lambda, \lambda \in P\operatorname{cf}\langle\lambda_i: i < \delta\rangle$ (*see below*). *If* $\lambda \in P\operatorname{Sc}_D\langle\lambda_i: i < \delta\rangle$, *we can strengthen the demand in* 3(A) *to* $\{i: |N \cap \lambda_i| \neq \varnothing \bmod D$.

DEFINITION 8. (A) $\lambda \in P\operatorname{Sc}_D\bar{\lambda}$ (λ is a possible scale for $\bar{\lambda}$), where $\bar{\lambda} = \langle\lambda_i: i < \delta\rangle, D$ a filter over $\delta, D \supseteq D(\delta) = \{A \subseteq \delta: \delta - A \text{ bounded}\}$, *if* λ, λ_i are regular cardinals or 1 and there are functions $f_\alpha(\alpha < \lambda)$ exemplifying it, i.e.

(a) $f_\alpha(i) < \lambda_i$ for $i < \delta$, and $\operatorname{Dom} f_\alpha = \delta$ (that is $f_\alpha \in \Pi_{i<\delta}\lambda_i$),

(b) $f_\alpha \leqq_D f_\beta$ for $\alpha < \beta$ (this means that $\{i: f_\alpha(i) \leqq f_\beta(i)\} \in D$),

(c) we cannot define f_λ satisfying (a) and (b).

(B) $\lambda \in P\operatorname{cf}\bar{\lambda}$ iff $\lambda \in P\operatorname{Sc}_D\bar{\lambda}$ for some ultrafilter D over δ.

(C) $\lambda \in P\operatorname{Sc}\bar{\lambda}$ if $\lambda \in P\operatorname{Sc}_{D(\delta)}\bar{\lambda}$

(D) $\bar{\lambda}$ is D trivial if $\{i: \lambda_i = 1\} \in D$; we always assume $\bar{\lambda}$ is not D-trivial.

OBSERVATION 9. (A) If $\lambda \in P\operatorname{Sc}_D\bar{\lambda}, \bar{\lambda} = \langle\lambda_i: i < \delta\rangle, 2^{|\delta|} < \lambda$, then $\lambda \in P\operatorname{cf}\bar{\lambda}$.

(B) $\lambda \in P\operatorname{Sc}_D\langle\lambda_i: i < \delta\rangle$ is equivalent to $\lambda = \operatorname{cf}[\Pi_{i<\delta}\lambda_i/D]$, for D an ultrafilter.

(C) Suppose $h: \delta^1 \to \delta^2, h_1: \delta^2 \to \delta^1, D_l$ a filter over δ^l,

$$\{i < \delta^1: \lambda_i \geqq \mu_{h(i)}\} \in D_1, A \in D_2 \Rightarrow \{i: h(i) \in A\} \in D_1,$$

$$\{j: hh_1(j) = j, \lambda_{h_1(j)} = \mu_j\} \in D_2$$

and $\delta^2 - A \notin D_2 \Rightarrow \delta^1 - \{h_1(j): j \in A\} \notin D_1$. Then $\mu \in P\operatorname{Sc}_{D_2}\langle\mu_j: j < \delta^2\rangle$ implies $\mu \in P\operatorname{Sc}_{D_1}\langle\lambda_i: i < \delta^1\rangle$.

(D) $\lambda \to \{\lambda_i: i < \delta\}[\delta]$ if $\lambda \in P\operatorname{Sc}_D\langle\lambda_i: i < \delta\rangle$.

PROOF OF LEMMA 7. (A), (B), (C). Immediate.

(D) Let M, N be as in Definition 3 (so $\lambda, \{\lambda_i: i < \delta\} \in N, \delta + 1 \subseteq N$). W.l.o.g. $\langle\lambda_i: i < \delta\rangle \in N, D \in N$ (by 4C); so there is $\langle f_\alpha: \alpha < \lambda\rangle \in N$ exemplifying $\lambda \in P\operatorname{Sc}_D\langle\lambda_i: i < \delta\rangle$. As $\delta + 1 \subseteq N, \lambda_i \in N$ for each i. If for each i $|N \cap \lambda_i| < \lambda_i$, then $A_i = \{f_\alpha(i): \alpha \in N \cap \lambda\}$ is a subset of λ_i of cardinality $< \lambda_i$, so by λ_i's regularity it has an upper bound $< \lambda_i$ which we call $f_\lambda(\alpha)$. It follows that for $\alpha \in N\, f_\alpha <_{D(\delta)} f_\lambda$ hence $f_\alpha <_D f_\lambda$: as $|N \cap \lambda| = \lambda$, and $<_D$ is transitive $f_\alpha <_D f_\lambda$ for each $\alpha < \lambda$; a contradiction.

Now we shall prove some cases of $\lambda \in P\,\mathrm{Sc}\,\bar{\lambda}$.

LEMMA 10. (A) *Let λ_i $(i < \delta)$ be increasing, $\delta < \lambda_* = \Sigma_{i<\delta}\lambda_i$, each λ_i a successor (at least for i limit or for an unbounded set of i's), then for any $f_\alpha \in \Pi_{i<\delta}\lambda_i$ $(\alpha < \lambda_*)$ there is an upper bound in $\Pi_{i<\delta}\lambda_i/D(\delta)$. Hence $\lambda \in P\,\mathrm{cf}_D\langle\lambda_i: i < \delta\rangle$ implies $\lambda > \lambda_*$.*

(B) *$\lambda \in P\,\mathrm{cf}_D\langle\lambda_i: i < \delta\rangle$ implies $\lambda \leqq \Pi_{i<\delta}\lambda_i$ (as cardinals).*

(C) *For every $\bar{\lambda}, D$, for some λ, $\lambda \in P\,\mathrm{Sc}_D\langle\lambda_i: i < \delta\rangle$.*

(D) *If $|\Pi_{i<\delta}\lambda_i/D| = \lambda_*^+$, $D \supseteq D(\delta)$ and the assumption of (A) holds then $\lambda_*^+ \in P\,\mathrm{Sc}_D\bar{\lambda}$.*

PROOF. Immediate (in (A) choose f such that $|\alpha|^+ < \lambda_i$ implies $f_\alpha(i) < f(i)$).

LEMMA 11. *Suppose $\bar{\lambda} = \langle\lambda_i: i < \kappa\rangle$, κ regular $< \lambda_* = \Sigma_{i<\kappa}\lambda_i$, λ_i is increasing.*

(A) *If $\lambda \in P\,\mathrm{Sc}_D\bar{\lambda}$, $\lambda_* < \mu < \lambda$, μ regular, D $\aleph_1$-complete or $2^\kappa < \mu$ then $\mu \in P\,\mathrm{Sc}_D\langle\lambda'_i: i < \kappa\rangle$ for some $\lambda'_i \leqq \lambda_i$, $\langle\lambda'_i: i < \kappa\rangle$ is not D-trivial.*

(B) *In (A), instead of $\lambda \in P\,\mathrm{Sc}_D\bar{\lambda}$ it suffices to assume: in $\Pi_{i<\kappa}\lambda_i/D$ there is a $<_D$-increasing sequence of length μ (or even $\leqq_D$-increasing, if it is not eventually constant by $=_D$).*

(C) *Note that in (A) and (B) if $\lambda_i^{\kappa} < \lambda_* \leqq \mu$ (for every i) then $\Sigma_{i<\kappa}\lambda'_i = \lambda_*$.*

(D) *If $\kappa > \aleph_0$ or $2^{\aleph_0} \leqq \lambda_*$ then $\mu = \lambda_*^+$ satisfies the requirement on μ in (A) for $D = D(\delta)$. (In the first case D is $\aleph_1$-complete and in the second $2^\kappa < \mu$.)*

PROOF. (A) follows from (B).

(B) Let f_α $(\alpha < \mu)$ be $<_D$-increasing (in $\Pi_{i<\kappa}\lambda_i/D$) s.t. $(\forall\alpha < \mu)(\exists\beta < \mu)(\alpha < \beta \wedge \neg f_\alpha =_D f_\beta)$. If they would exemplify $\mu \in P\,\mathrm{Sc}_D\bar{\lambda}$, we finish. Otherwise we shall show that

(*) there is $f \in \Pi_{i<\kappa}\lambda_i/D$ such that $f_\alpha \leqq_D f$, for $\alpha < \mu$, but for no g is $f_\alpha \leqq_D g <_D f$ for every $\alpha < \mu$.

Now (*) is sufficient, for let $\lambda'_i = \mathrm{cf}\, f(i)$, $A_i \subseteq f(i)$ a close unbounded set of order-type $\mathrm{cf}\, f(i)$, $A_i = \{\alpha(i,j): j < \lambda'_i\}$ ($\alpha(i,j)$ increasing with j) (if $f(i)$ is a successor ordinal $\lambda'_i = 1$).

Let $f'_\alpha(i) = \min\{j: \alpha(i,j) \geqq f_\alpha(i)\}$, then f'_α $(\alpha < \mu)$ exemplify $\mu \in P\,\mathrm{Sc}_D\langle\lambda'_i: i < \kappa\rangle$, $\langle\lambda'_i: i < \kappa\rangle$ is not D-trivial, as otherwise we find g contradicting (*).

Let us prove (*).

Case (i). D is $\aleph_1$-complete.

In this case $<_D$ is well-founded, as we assume there is $f \in \Pi_{i<\kappa}\lambda_i/D$, $f_\alpha \leqq_D f$ for every $\alpha < \mu$, there is one as required.

Case (ii). $2^\kappa < \mu$.

It is well known that there is no decreasing sequence of length $(2^\kappa)^+$ in $<_D$. So define by induction on γ $f^\gamma \in \Pi_{i<\kappa}\lambda_i$, such that $\beta < \gamma \Rightarrow f^\gamma <_D f^\beta$, and $\alpha < \mu \Rightarrow f_\alpha \leqq_D f^\gamma$. Now f^0 exists by an assumption in the beginning of the proof. So there is a first γ_0 for which f^{γ_0} is not defined. We shall now prove γ_0 is a successor so $f^{\gamma_0 - 1}$ is as required. As mentioned above $\gamma_0 < (2^\kappa)^+$. Let $P_i = \{f^\gamma(i);\ \gamma < \gamma_0\} \subseteq \lambda_i$, so $|P_i| \leqq 2^\kappa$. Let $(\Pi_{i<\kappa}\lambda_i/D, \leqq, P) = \Pi_i(\lambda_i, \leqq_D, P_i)/D$ so $|P| \leqq \Pi_{i<\kappa}|P_i| \leqq 2^\kappa$. Now $2^\kappa < \mu$, μ regular so for some $\alpha_0 < \mu$, for every $a \in P$, and $\alpha_0 \leqq \alpha < \mu$, $f_{\alpha_0} \leqq_D a \Leftrightarrow f_\alpha \leqq_D a$. Now

$$(\lambda_i, \leqq, P_i) \models (\forall x)[(\exists z)(P(z) \wedge x \leqq z) \rightarrow (\exists y)[(P(y) \wedge x \leqq y) \wedge (\forall z)(P(z) \wedge x \leqq z \rightarrow y \leqq z)]].$$

This is a Horn sentence, so $(\Pi_{i<\kappa}\lambda_i/D, \leqq_D, P)$ satisfies it, so taking f_{α_0} for x the antecedent holds ($z = f^0$) so we get f for y. So $f_{\alpha_0} \leqq_D f$ hence for every α $f_\alpha \leqq_D f$ by the choice of f_{α_0}; also $f \leqq_D f^\gamma$ as $(\Pi_{i<\kappa}\lambda_i/D, \leqq_D, P) \models P(f^\gamma) \wedge f_{\alpha(0)} \leqq_D f^\gamma$. Clearly f is as required.

(C), (D) left to the reader.

Conclusion 12. For $\aleph_\delta$ singular, D an ultrafilter over $\operatorname{cf}\delta$, in $(\omega_\delta, <)^{\operatorname{cf}\delta}/D$ there is no increasing sequence of length $\aleph_\gamma$ where $\gamma = (|\delta|^{\operatorname{cf}\delta}/D)^+$.

Proof. Otherwise for every $\beta < \gamma$, β successor, $\beta > \delta$ there are $\alpha(\beta, i) < \delta$ ($i < \operatorname{cf}\delta$) such that $\operatorname{cf}[\Pi_{i<\delta}(\omega_{\alpha(\beta,i)}, <)/D] = \aleph_\beta$ (by 11A, 9A) but the number of possible $\langle \alpha(\beta, i) : i < \operatorname{cf}\delta \rangle$ is $\leqq |\delta|^{\operatorname{cf}\delta}/D$, contradiction.

This has relation to Galvin and Hajnal [2], but 12 is applicable when $\operatorname{cf}\delta = \aleph_0$ too. In fact

Claim 13. If $\aleph_\delta$ is singular, $\operatorname{cf}\delta > \aleph_0$, $\mu \leqq \aleph_\delta^{\operatorname{cf}\delta}$ regular, $(\forall \alpha < \delta)(\forall k < \operatorname{cf}\delta)\aleph_\alpha^k < \aleph_\delta$ then for some $\alpha(i) < \delta$, $\mu \in P\operatorname{Sc}\langle \aleph_{\alpha(i)} : i < \delta \rangle$.

If $\beta(i)$ ($i < \operatorname{cf}\delta$) are increasing and continuous with limit δ, for $\mu = \aleph_{\delta+1}$ we can choose $\alpha(i) = \beta(i) + 1$ provided that $\Pi_{i<j}\aleph_{\alpha(i)} \leqq \aleph_{\alpha(j)}$.

We can now apply our theorems.

Conclusions 14. (A) $\operatorname{Jn}(\aleph_{\omega+1})$ if $2^{\aleph_0} \leqq \aleph_{\omega+1}$.

(B) If $(\forall\lambda)$ $(\mathrm{cf}\,\lambda > \aleph_0 \to \lambda^{\aleph_0} = \lambda)$ and there is no weakly inaccessible cardinal then $(\forall\lambda)$ $\mathrm{Jn}(\lambda^+)$.

PROOF. (A) First note that for any non-principal ultrafilter D over ω, $\aleph_{\omega+1} \in P\,\mathrm{Sc}_D\langle\aleph_{n(k)}: k<\omega\rangle$ (for some $n(k)<\omega$) (if $2^{\aleph_0} = \aleph_{\omega+1}$, by 10(D), otherwise for some λ, $\lambda \in P\,\mathrm{Sc}_D\langle\aleph_n: n<\omega\rangle$; by 10(A) $\lambda > \aleph_\omega$, by 11A $\aleph_{\omega+1} \in P\,\mathrm{Sc}_D\langle\aleph_{n(k)}: k<\omega\rangle$ for some $n(k)$). For a given $m<\omega$, we can assume $n(k) \geqq m$ (as $\{k: n(k)<m\} \notin D$), by 7(D) $\aleph_{\omega+1} \to \{\aleph_{n(k)}: k<\omega\}[\aleph_0]$. As $\aleph_n \to \aleph_m[\aleph_0]$ for $n \geqq m$ (by 7A), by 6(A) $\aleph_{\omega+1} \to \aleph_m[\aleph_0]$. So by 5(A) $\mathrm{Jn}(\aleph_{\omega+1})$.

(B) Left to the reader.

CONCLUSION 15. $\mathrm{Jn}(2^{\aleph_0})$ if $2^{\aleph_0} = \aleph_{\alpha+1}$, $\alpha < \omega_1$.

PROOF. Let $\beta \leqq \alpha$ and we shall prove $\aleph_{\alpha+1} \to \aleph_{\beta+1}[\aleph_0]$ (this is sufficient by 5A). We define increasing $\beta(i) \leqq \alpha+1$, and $S_i \subseteq \{\aleph_{\beta(j)}: j<i\}$, $\beta(0)=\beta+1$, each $\beta(i)$ is a successor, to satisfy 6(B). For $i=0, \beta(0)=\beta+1, \beta(i+1)=\beta(i)+1, S_{i+1}=\{\aleph_{\beta(i)}\}$. For i limit of cofinality ω let $i_n < i$ be increasing with limit i, $S_i = \{\aleph_{\beta(i_n)}: n<\omega\}$, and we choose a successor $\beta(i) > \bigcup_n \beta(i_n)$, $\beta(i) \leqq \alpha+1$ such that $\aleph_{\beta(i)} \to S_i[\aleph_0]$; we can do it by 10C and 10A, B. By 6B $\aleph_{\alpha+1} \to \aleph_{\beta+1}[\aleph_0]$, thus we finish.

LEMMA 16. *If $\lambda \to \mu^+[\aleph_0]$ for every μ, $\lambda_0 \leqq \mu < \lambda$ and $N < M_\lambda$, $\|N\| = \lambda$ then:*

(A) *If $\lambda_0 \leqq \mu \leqq \lambda$ then $\mu \in N$ and $|\mu \cap N| = \mu$ (so $\lambda \to \mu[\aleph_0]$).*

(B) *For every $a \in \lambda$ there is b such that $a \in b \in N$, and $|b| < \lambda_0$.*

(C) *If $\lambda^{\lambda_0} = \lambda$ then* $\mathrm{Jn}(\lambda)$.

PROOF. (A) Like 5(A) (notice we can assume λ_0 is minimal with such properties, hence definable in M_λ).

(B) Let μ be a minimal cardinal such that for some b_μ, $|b_\mu| \leqq \mu$, $a \in b_\mu \in N$. Now $\mu \leqq \lambda$ as we can choose $b_\lambda = \lambda$.

Let us prove $\mu < \lambda_0$; otherwise as $b_\mu \in N$ also $\mu = |b_\mu| \in N$, so in N there is a function f from μ onto b_μ. We know by 15(A) that $|\mu \cap N| = \mu$, so $N \cap \mu$ is unbounded in μ, so there is $\alpha < \mu, \alpha \in N$ such that $a \in \{f(\beta); \beta < \alpha\}$. Now $b' = \{f(\beta): \beta < \alpha\} \in N$ contradicts μ's minimality.

(C) It suffices to prove $\lambda \subseteq N$, so let $a \in \lambda$. By 15(B) there is $b \in N, |b| \leqq \lambda_0, a \in b$, and as $\lambda_0 \in N$ we can assume $|b| = \lambda_0$. As $|N \cap \lambda| = \lambda$ there is a set $A \subseteq \lambda \cap N, |A| = \lambda_0$ and necessarily $A \in M^1_\lambda$ but possibly $A \notin N$. Let $F^* \in N$ be a function from λ onto $\{B \subseteq \lambda: |B| = \lambda_0\}$; so for some $i, j < \lambda, F^*(i) = A, F^*(j) = b$. By 15(A) there is $C \in N, |C| \leqq \lambda_0$ such that $i, j \in C$. $\{F^*(\alpha): \alpha \in C\}$ is a family of $\leqq \lambda_0$ sets each of power exactly λ_0. So there is a function $g \in N$, $\operatorname{Dom} g = \bigcup_{\alpha \in C} F^*(\alpha)$, such that for every $\alpha \in C$, $\{g(x): x \in F^*(\alpha)\} = \operatorname{Dom} g$ (clearly $|\operatorname{Dom} g| = \lambda_0$).

This holds for $\alpha = i$, but $g \in N$, $A = F^*(i) \subseteq N$; so $\operatorname{Dom} g \subseteq N$, but $a \in b = F^*(j)$, $j \in C$ so $a \in N$.

Conclusion 17. Suppose $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma+1}$, then $\operatorname{Jn}(2^{\aleph_\alpha})$ if (A) or (B) or (C):

(A) $\gamma < \omega_1$,

(B) $2^{\aleph_\alpha} \to \mu[\aleph_0]$ for every $\mu \leqq |\gamma|$,

(C) $\beta < \alpha \Rightarrow 2^{\aleph_\beta} < 2^{\aleph_\alpha}$, and $\operatorname{Jn}(\aleph_\alpha)$ and $\gamma < \aleph_{\alpha+1}$.

Proof. Similar to 14.

References

1. C. C. Chang and H. J. Keisler, *Model Theory*, North Holland Publ. Co., Amsterdam, 1973.
2. F. Galvin and A. Hajnal, *Inequalities for cardinal powers*, Ann. of Math. **101** (1975), 489–491.
3. S. Shelah, *Notes in combinatorial set theory*, Israel J. Math. **14** (1973), 262–277.

Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel